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On asymptotic failure rates in bivariate frailty competing risks models

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Abstract

A bivariate competing risks problem is considered for a rather general class of survival models. The lifetime distribution of each component is indexed by a frailty parameter. Under the assumption of conditional independence of components the correlated frailty model is considered. The explicit asymptotic formula for the mixture failure rate of a system is derived. It is proved that asymptotically, as $t \to \infty$, the remaining lifetimes of components tend to be independent in the defined sense. Some simple examples are discussed.

1. Introduction

It is well known that mixtures of distributions are a convenient tool for analyzing univariate frailty models. As monotonicity properties of the mixture failure rate can differ dramatically from those of the baseline failure rate, this topic was thoroughly investigated in the literature (see Badia et al. (2001), Block et al. (1993), Finkelstein and Esaulova (2001) and Lynch (1999), to name a few). Considerable attention was also paid to the asymptotic behavior of mixture failure rates (Block et al., 2003; Finkelstein and Esaulova, 2006; Shaked and Spizzichino, 2001).

In our paper (Finkelstein and Esaulova, 2006) a general class of lifetime models with frailties were considered. A basic model for $F(t,z)$ — an absolutely continuous cumulative distribution function (Cdf) of a lifetime random variable T, was defined as

$$\Lambda(t,z) = A(z\phi(t)) + \psi(t), \tag{1}$$

where $\Lambda(t,z) = \int_0^t \lambda(u,z)\mathrm{d}u$ is the corresponding cumulative failure rate and z is a realization of frailty Z. The general assumptions on the functions involved were rather natural: $A(s)$, $\phi(t)$ and $\psi(t)$ are differentiable, the right-hand side of (1) is non-decreasing in t and increases to infinity as $t \to \infty$, and $A(z\phi(0)) + \psi(0) = 0$.

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Many of the models popular in reliability, survival analysis and risk analysis (proportional hazards (PH), additive hazards (AH) and accelerated life (ALM) models), are obviously special cases of (1):

PH (multiplicative) model:

Let

$$A(u) \equiv u, \qquad \phi(t) = \Lambda(t), \qquad \psi(t) = 0.$$

Then

$$\lambda(t, z) = z\lambda(t), \qquad \Lambda(t, z) = z\Lambda(t). \tag{2}$$

Accelerated life model:

Let

$$A(u) \equiv \Lambda(u), \qquad \phi(t) = t, \qquad \psi(t) = 0.$$

Then

$$\Lambda(t, z) = \int_0^{tz} \lambda(u)\mathrm{d}u = \Lambda(tz), \qquad \lambda(t, z) = z\lambda(tz). \tag{3}$$

AH model:

Let

$$A(u) \equiv u, \qquad \phi(t) = t, \qquad \psi(t) \text{ is increasing}, \qquad \psi(0) = 0.$$

Then

$$\lambda(t, z) = z + \psi'(t), \qquad \Lambda(t, z) = zt + \psi(t). \tag{4}$$

Under the stated assumptions and using some additional technical conditions for the pdf of the frailty Z, we derived exact asymptotic relations for the corresponding mixture failure rate $\lambda_m(t)$ as $t \to \infty$ (Finkelstein and Esaulova, 2006).

In the current study we use and develop asymptotic methodology employed for the univariate case for analyzing the behavior of failure rates in the competing risk setting with a bivariate frailty.

Section 2 is devoted to basic definitions and some supplementary simple non-asymptotic properties of mixture failure rates with independent frailties.

In Section 3 we obtain explicit asymptotic results, which above, all show, that even in the case of correlated frailty the components' remaining lifetimes can be considered as 'asymptotically independent' in the defined sense.

In Section 4 we provide some relevant examples and discuss the restrictions of our assumptions. It is worth noting that the generalization of our results to the multivariate case when $n > 2$ is rather straightforward.

2. Bivariate frailty and competing risks

Assume that risks are dependent only via the bivariate frailty (Z_1, Z_2). To construct the corresponding competing risks model consider firstly a system of two statistically independent components in series with lifetimes $T_1 \geq 0$ and $T_2 \geq 0$. The Cdf function of this system is

$$F_s(t) = 1 - \bar{F}_1(t)\bar{F}_2(t),$$

where $F_1(t)$ and $F_2(t)$ are the Cdfs of the lifetime random variables T_1 and T_2 respectively ($\bar{F}_i(t) \equiv 1 - F_i(t)$).

Assume now that $F_i(t),\ i = 1, 2$ are indexed by random variables Z_i in the following conventional sense:

$$P(T_i \leq t \mid Z_i = z) \equiv P(T_i \leq t \mid z) = F_i(t, z), \quad i = 1, 2$$

and that the pdfs $f_i(t, z)$ exist. Then the corresponding failure rates $\lambda_i(t, z)$ are $f_i(t, z)/\bar{F}_i(t, z)$.

Let $Z_i,\ i = 1, 2$ be interpreted as non-negative random variables with supports in $[a_i, b_i],\ a_1 \geq 0,\ b_i \leq \infty$ and the pdf $\pi_i(z)$.

A mixture Cdf for the ith component is defined by

$$F_{m,i}(t) = \int_{a_i}^{b_i} F_i(t,z)\pi_i(z)\mathrm{d}z, \quad i = 1, 2. \tag{5}$$

The corresponding mixture failure rate is:

$$\lambda_{m,i}(t) = \frac{\int_{a_i}^{b_i} f_i(t,z)\pi_i(z)\mathrm{d}z}{\int_{a_i}^{b_i} \bar{F}_i(t,z)\pi_i(z)\mathrm{d}z} = \int_{a_i}^{b_i} \lambda_i(t,z)\pi(z \mid t)\mathrm{d}z, \tag{6}$$

where the conditional pdf (on the condition that $T_i > t$):

$$\pi_i(z \mid t) = \pi_i(z)\frac{\bar{F}_i(t,z)}{\int_{a_i}^{b_i} \bar{F}_i(t,z)\pi_i(z)\mathrm{d}z}. \tag{7}$$

Assume that the components of our system are conditionally independent given $Z_1 = z_1$, $Z_2 = z_2$. Then the Cdf of the system is:

$$F_s(t, z_1, z_2) = 1 - \bar{F}_1(t, z_1)\bar{F}_2(t, z_2) \tag{8}$$

and the corresponding probability density function is

$$f_s(t, z_1, z_2) = f_1(t, z_1)\bar{F}_2(t, z_2) + f_2(t, z_2)\bar{F}_1(t, z_1). \tag{9}$$

The mixture failure rate of the system in this case is defined as

$$\begin{aligned}\lambda_{m,s}(t) &= \frac{\int_{a_2}^{b_2}\int_{a_1}^{b_1} f_s(t,z_1,z_2)\pi(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2}{\int_{a_2}^{b_2}\int_{a_1}^{b_1} \bar{F}_s(t,z_1,z_2)\pi(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2} \\ &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} \lambda_s(t,z_1,z_2)\pi(z_1,z_2 \mid t)\mathrm{d}z_1\mathrm{d}z_2,\end{aligned} \tag{10}$$

where

$$\pi(z_1, z_2 \mid t) = \pi(z_1,z_2)\frac{\bar{F}_s(t,z_1,z_2)}{\int_{a_2}^{b_2}\int_{a_1}^{b_1} \bar{F}_s(t,z_1,z_2)\pi(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2}, \tag{11}$$

and $\pi(z_1, z_2)$ is the bivariate joint probability density function of Z_1 and Z_2. It is clear that for our series system, defined by (8):

$$\lambda_s(t, z_1, z_2) = \lambda_1(t, z_1) + \lambda_2(t, z_2). \tag{12}$$

It is clear also that if Z_1 and Z_2 are independent, which means

$$\pi(z_1, z_2) = \pi_1(z_1)\pi_2(z_2)$$

for some densities $\pi_1(z_1)$ and $\pi_2(z_2)$; then

$$\pi(z_1, z_2|t) = \pi_1(z_1|t)\pi_2(z_2|t),$$

which can be easily seen using definitions (7) and (11):

$$\begin{aligned}\pi(z_1, z_2|t) &= \pi_1(z_1)\pi_2(z_2)\frac{\bar{F}_1(t,z_1)\bar{F}_2(t,z_2)}{\int_{a_2}^{b_2}\int_{a_1}^{b_1} \bar{F}_1(t,z_1)\bar{F}_2(t,z_2)\pi_1(z_1)\pi_2(z_2)\mathrm{d}z_1\mathrm{d}z_2} \\ &= \frac{\pi_1(z_1)\bar{F}_1(t,z_1)\cdot\pi_2(z_2)\bar{F}_2(t,z_2)}{\int_{a_1}^{b_1}\bar{F}_1(t,z_1)\pi_1(z_1)\mathrm{d}z_1\cdot\int_{a_2}^{b_2}\bar{F}_2(t,z_2)\pi_2(z_2)\mathrm{d}z_2} \\ &= \pi_1(z_1|t)\pi_2(z_2|t).\end{aligned}$$

Using Eqs. (10) and (12):

$$\begin{aligned}\lambda_{m,s}(t) &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} \lambda_s(t, z_1, z_2)\pi(z_1, z_2 \mid t)\mathrm{d}z_1\mathrm{d}z_2 \\ &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} [\lambda_1(t, z_1) + \lambda_2(t, z_2)]\,\pi_1(z_1|t)\pi_2(z_2|t)\mathrm{d}z_1\mathrm{d}z_2 \\ &= \int_{a_1}^{b_1} \lambda_1(t, z_1)\pi_1(z_1|t)\mathrm{d}z_1 + \int_{a_2}^{b_2} \lambda_2(t, z_2)\pi_2(z_2|t)\mathrm{d}z_2 \\ &= \lambda_{m,1}(t) + \lambda_{m,2}(t). \end{aligned} \tag{13}$$

Hence, when the components of the system are conditionally independent and Z_1 and Z_2 are independent, the mixture failure rate of the system is the sum of mixture failure rates of individual components.

It is worth noting that Eq. (13) does not hold for the case of shared frailty, when $Z_1 \equiv Z_2 \equiv Z$, as

$$\begin{aligned}\lambda_{ms}(t) &= \frac{\int_a^b f_s(t, z)\pi(z)\mathrm{d}z}{\int_a^b \bar{F}_s(t, z)\pi(z)\mathrm{d}z} \\ &= \frac{\int_a^b f_1(t, z)\bar{F}_2(t, z)\pi(z)\mathrm{d}z}{\int_a^b \bar{F}_s(t, z)\pi(z)\mathrm{d}z} + \frac{\int_a^b f_2(t, z)\bar{F}_1(t, z)\pi(z)\mathrm{d}z}{\int_a^b \bar{F}_s(t, z)\pi(z)\mathrm{d}z} \\ &= \frac{\int_a^b \lambda_1(t, z)\bar{F}_s(t, z)\pi(z)\mathrm{d}z}{\int_a^b \bar{F}_s(t, z)\pi(z)\mathrm{d}z} + \frac{\int_a^b \lambda_2(t, z)\bar{F}_s(t, z)\pi(z)\mathrm{d}z}{\int_a^b \bar{F}_s(t, z)\pi(z)\mathrm{d}z}\end{aligned}$$

is not equal to $\lambda_{1m}(t) + \lambda_{2m}(t)$.

In the next section we shall study the asymptotic behavior of mixture failure rates for the most interesting case of correlated frailty.

3. The main result

Assume that lifetimes of both components belong to the class defined by relation (1). For simplicity let the unimportant additive term be equal to zero. The corresponding survival functions for the components are

$$\bar{F}_i(t, z_i) = \mathrm{e}^{-A_i(z_i\phi_i(t))}, \quad i = 1, 2. \tag{14}$$

The following result is obtained:

Theorem 1. *Let the corresponding survival functions in the competitive risks model (8) be defined by Eq. (14).*

Suppose that the mixing variables Z_1 and Z_2 have a joint probability density function $\pi(z_1, z_2)$, which is defined in $[0, b_1] \times [0, b_2]$, $0 < b_1, b_2 \le \infty$.

Let the following properties hold:

(a) $\pi(z_1, z_2) = z_1^{\alpha_1} z_2^{\alpha_2}\pi_0(z_1, z_2)$, *where* $\alpha_1, \alpha_2 > -1$.

(b) $\pi_0(z_1, z_2)$ *is continuous at* $(0, 0)$, $\pi_0(0, 0) \neq 0$.

(c) $A_i(s),\ i = 1, 2$ *are positive ultimately increasing differentiable functions,*

$$\int_0^\infty \mathrm{e}^{-A_i(s)} s^{\alpha_i}\mathrm{d}s < \infty.$$

Assume finally that $\phi_1(t), \phi_2(t) \to \infty$ as $t \to \infty$.

Then

$$\lambda_{m,s}(t) \sim (\alpha_1 + 1)\frac{\phi_1'(t)}{\phi_1(t)} + (\alpha_2 + 1)\frac{\phi_2'(t)}{\phi_2(t)}. \tag{15}$$

By the sign $\sim$ we, as usual, denote the asymptotic equivalence: $g_1(t) \sim g_2(t)$ as $t \to \infty$ means that $g_1(t)/g_2(t) \to 1$ as $t \to \infty$.

Remark. It follows from the additive nature of the left-hand side of (15) and the corresponding result for the univariate case (Finkelstein and Esaulova, 2006) that the asymptotic mixture failure rate in our model can be viewed as the sum of univariate mixture failure rates of each component with its own independent frailty. Therefore, taking into account relation (13), it is easy to arrive at the following important interpretation of the theorem:

Under certain assumptions the asymptotic mixture failure rate in the correlated frailty model with conditionally independent components is equivalent to the asymptotic mixture failure rate in the independent frailty model.

This can be also considered as some asymptotic independence of remaining lifetimes of our components in the correlated frailty model.

Proof. We start our proof with the following supplementary lemma:

Lemma 1. *Let $g(z_1, z_2)$ be a non-negative integrable function in $[0, \infty)^2$. Let $h(z_1, z_2)$ be a non-negative locally integrable function defined in $[0, \infty)^2$, such that it is bounded everywhere and continuous at the origin.*

Then, as $t_1 \to \infty, t_2 \to \infty$:

$$t_1 t_2 \int_0^\infty \int_0^\infty g(t_1 z_1, t_2 z_2) h(z_1, z_2) \mathrm{d}z_1 \mathrm{d}z_2 \to h(0,0) \int_0^\infty \int_0^\infty g(z_1, z_2) \mathrm{d}z_1 \mathrm{d}z_2.$$

Proof. The proof is rather straightforward:

$$t_1 t_2 \int_0^\infty \int_0^\infty g(t_1 z_1, t_2 z_2) h(z_1, z_2) \mathrm{d}z_1 \mathrm{d}z_2 = \int_0^\infty \int_0^\infty g(z_1, z_2) h\left(\frac{z_1}{t_1}, \frac{z_2}{t_2}\right) \mathrm{d}z_1 \mathrm{d}z_2.$$

Indeed, $h(z_1, z_2)$ is bounded; assume that it is bounded by some M. The function $g(z_1, z_2)$ is integrable, then for any $\epsilon > 0$ there is a finite $b > 0$, such that

$$\iint_{[0,\infty)^2 - [0,b]^2} g(z_1, z_2) \mathrm{d}z_1 \mathrm{d}z_2 < \epsilon.$$

Then

$$\left| \int_0^\infty \int_0^\infty g(z_1, z_2) \left[h\left(\frac{z_1}{t_1}, \frac{z_2}{t_2}\right) - h(0,0) \right] \mathrm{d}z_1 \mathrm{d}z_2 \right| \le \int_0^b \int_0^b g(z_1, z_2) \left| h\left(\frac{z_1}{t_1}, \frac{z_2}{t_2}\right) - h(0,0) \right| \mathrm{d}z_1 \mathrm{d}z_2 + 2M \iint_{[0,\infty)^2 - [0,b]^2} g(z_1, z_2) \mathrm{d}z_1 \mathrm{d}z_2.$$

The first double integral tends to zero since $h(z_1, z_2)$ is continuous at (0, 0), and the second can be made arbitrarily small. $\square$

Now we can proceed with the proof of the theorem. Substituting (8) and (9) into (10) we get

$$\lambda_{m,s}(t) = \frac{\int_0^{b_1} \int_0^{b_2} f_1(t, z_1) \bar{F}_2(t, z_2) \pi(z_1, z_2) \mathrm{d}z_2 \mathrm{d}z_1}{\int_0^{b_1} \int_0^{b_2} \bar{F}_1(t, z_1) \bar{F}_2(t, z_2) \pi(z_1, z_2) \mathrm{d}z_2 \mathrm{d}z_1} + \frac{\int_0^{b_2} \int_0^{b_1} f_2(t, z_2) \bar{F}_1(t, z_1) \pi(z_1, z_2) \mathrm{d}z_1 \mathrm{d}z_2}{\int_0^{b_2} \int_0^{b_1} \bar{F}_2(t, z_1) \bar{F}_1(t, z_1) \pi(z_1, z_2)}. \quad (16)$$

Denote the first term on the right-hand side by $\lambda^1_{m,s}(t)$ and the second one by $\lambda^2_{m,s}(t)$. Then

$$\lambda_{m,s}(t) = \lambda^1_{m,s}(t) + \lambda^2_{m,s}(t).$$

Consider $\lambda^1_{m,s}(t)$ and $\lambda^2_{m,s}(t)$ separately. The probability density function of T_1 is

$$f_1(t, z_1) = A'_1(z_1 \phi_1(t)) z_1 \phi'_1(t) \mathrm{e}^{-A_1(z_1 \phi_1(t))} \quad (17)$$

and

$$\lambda^1_{m,s}(t) = \frac{\int_0^{b_1} \int_0^{b_2} A'_1(z_1 \phi_1(t)) z_1 \phi'_1(t) \mathrm{e}^{-A_1(z_1 \phi_1(t)) - A_2(z_2 \phi_2(t))} \pi(z_1, z_2) \mathrm{d}z_2 \mathrm{d}z_1}{\int_0^{b_1} \int_0^{b_2} \mathrm{e}^{-A_1(z_1 \phi_1(t)) - A_2(z_2 \phi_2(t))} \pi(z_1, z_2) \mathrm{d}z_2 \mathrm{d}z_1}.$$

Applying the lemma to the numerator, we see that it is asymptotically equivalent to

$$\frac{\phi_1'(t)\pi_0(0,0)}{\phi_1(t)^{\alpha_1+2}\phi_2(t)^{\alpha_2+1}}\int_0^\infty A_1'(u)u^{\alpha_1+1}\mathrm{e}^{-A_1(u)}\mathrm{d}u\int_0^\infty s^{\alpha_2}\mathrm{e}^{-A_2(s)}\mathrm{d}s$$

and the denominator is equivalent to

$$\frac{\pi_0(0,0)}{\phi_1(t)^{\alpha_1+1}\phi_2(t)^{\alpha_2+1}}\int_0^\infty u^{\alpha_1}\mathrm{e}^{-A_1(u)}\mathrm{d}u\int_0^\infty s^{\alpha_2}\mathrm{e}^{-A_2(s)}\mathrm{d}s.$$

Hence,

$$\lambda_{m,s}^1(t)\sim\frac{\phi_1'(t)}{\phi_1(t)}\cdot\frac{\int_0^\infty A_1'(u)u^{\alpha_1+1}\mathrm{e}^{-A_1(u)}\mathrm{d}u}{\int_0^\infty u^{\alpha_1}\mathrm{e}^{-A_1(u)}\mathrm{d}u}. \tag{18}$$

Due to condition (c) of the theorem

$$\mathrm{e}^{-A(s)}s^{\alpha+1}\to 0\quad\text{as } s\to\infty. \tag{19}$$

Indeed, by the mean value theorem:

$$\int_s^{2s}\mathrm{e}^{-A(u)}u^\alpha\mathrm{d}u=s\mathrm{e}^{-A(s_1)}s_1^\alpha$$

for some $s\le s_1\le 2s$. The right-hand side tends to 0. For s larger than some s_0 we have $A(s_1)>A(s)$; thus, the left-hand side is smaller than $2^\alpha s^{\alpha+1}\mathrm{e}^{-A(s)}$, which leads to (19). Using it while integrating by parts, we get

$$\int_0^\infty A'(s)\mathrm{e}^{-A(s)}s^{\alpha+1}\mathrm{d}s=(\alpha+1)\int_0^\infty\mathrm{e}^{-A(s)}s^\alpha\mathrm{d}s. \tag{20}$$

Thus, from (18)

$$\lambda_{m,s}^1(t)\sim(\alpha_1+1)\frac{\phi_1'(t)}{\phi_1(t)}.$$

Similarly,

$$\lambda_{m,s}^2(t)\sim(\alpha_2+1)\frac{\phi_2'(t)}{\phi_2(t)}.\quad\square$$

4. Discussion

Assumptions (a) and (b) of the theorem impose certain restrictions on the mixing distribution. The corresponding conditions in the univariate case are satisfied for a wide class of distributions (admissible class), such as Gamma, Weibull, etc. (Finkelstein and Esaulova, 2006). In the bivariate case they obviously hold, at least, for all densities that are positive and continuous at the origin.

It is worth to interpret our results in terms of copulas, which can be helpful in analyzing the competing risks problems.

The following result, which defines simple sufficient conditions, is obvious and therefore its proof is omitted:

Corollary 1. *Assume that the bivariate mixing Cdf is given by the copula $C(u,v)$:*

$$\Pi(z_1,z_2)=C(\Pi_1(z_1),\Pi_2(z_2)),$$

*where $\Pi_1(z_1)$, $\Pi_2(z_2)$ are univariate Cdfs, which densities satisfy the following univariate conditions (*Finkelstein and Esaulova, 2006*):*

$$\pi_i(z)=z^{\alpha_i}\pi_{i,0}(z),\quad \alpha_i>-1,$$

where $\pi_{i,0}(z)$, $i=1,2$ are bounded in $[0,\infty)$, continuous and positive at $z=0$ (admissible class).

Then the bivariate conditions (a) and (b) of the theorem are satisfied, if $c(u, v) = \frac{\partial^2 C}{\partial u \partial v}(u, v)$ *can be represented as*

$$c(u, v) = u^{\gamma_1} v^{\gamma_2} c_0(u, v), \tag{21}$$

where $c_0(u, v)$ *is continuous and positive at* $(0, 0)$ *and* $\gamma_1, \gamma_2 \geq 0$.

Example. Farlie–Gumbel–Morgenstern copula. The corresponding mixing distribution is defined via the copula:

$$C(u, v) = uv(1 + \theta(1 - u)(1 - v)),$$

where $|\theta| \leq 1$, $u, v \in [0, 1]$. Since

$$\frac{\partial^2 C}{\partial u \partial v}(u, v) = 1 + \theta(1 - 2u)(1 - 2v)$$

is continuous at the origin and positive there if $\theta > -1$, the bivariate conditions hold when $-1 < \theta \leq 1$. Therefore, the results of the theorem hold if the univariate Cdfs belong to the admissible class.

Other mixing distributions that meet the conditions of the theorem are the Dirichlet distribution (Kotz et al., 2000, p. 485), the inverted Dirichlet distribution (Kotz et al., 2000, p. 491), some types of multivariate logistic distributions (Kotz et al., 2000, p. 551) and some types of special bivariate extreme value distributions (Kotz et al., 2000, p. 625).

There are also examples where conditions of the theorem do not hold. This happens, e.g., when the joint Cdf depends on $\max(z_1, z_2)$ and is not absolutely continuous. The widely used bivariate exponential distribution of Marshall and Olkin with the survival function

$$\bar{\Pi}(z_1, z_2) = \mathrm{e}^{-\gamma_1 z_1 - \gamma_2 z_2 - \gamma_{12} \max(z_1, z_2)}$$

is a relevant example. Some multivariate Weibull distributions also employ max functions and are not absolutely continuous at (0, 0). The corresponding examples can be also found in Kotz et al. (2000, p. 431).

Finally, in order to illustrate explicitly the main result of this paper given by the theorem, assume that the lifetimes of both components can be described by the PH model (2) and $\alpha_1 = \alpha_2 = 0$. Then, as $t \to \infty$, in accordance with (15)

$$\lambda_{m,s}(t) \sim \frac{\lambda_1(t)}{\int_0^t \lambda_1(u)\mathrm{d}u} + \frac{\lambda_2(t)}{\int_0^t \lambda_2(u)\mathrm{d}u},$$

which is a simple and easy interpretable asymptotic formula.

If the lifetimes of both components are described by the ALM model (3), then the asymptotics are surprisingly simple:

$$\lambda_{m,s}(t) \sim \frac{2}{t}.$$

Both of these formulas show that in this case ($\alpha_1 = \alpha_2 = 0$) the asymptotic behavior of the system mixture failure rate does not depend on the mixing distribution at all.

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